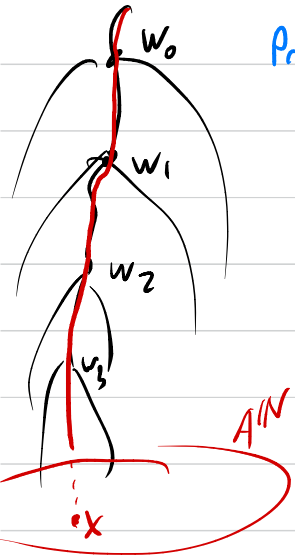
Metric Spaces and Topology
Lecture 7

Examples. ∘ For any set A, $A^{\mathbb{N}}$ is complete with the usual metric.

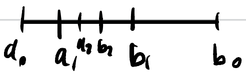


Proof. Recall that a ball in this space is clopen and equal to a cylinder $[w]$ for some $w \in A^{<\mathbb{N}}$.
Note that if $[w_1] \supseteq [w_2] \iff w_1 \subseteq w_2$.
Thus, a decreasing seq. (B_n) of balls is $B_n = [w_n]$ where $w_n \subseteq w_{n+1}$.
Suppose $\operatorname{diam}(B_n) \to 0$, so $w_n \subsetneq w_{n+1}$.
Then $\bigcup_n w_n$ is an infinite word, i.e. $= x \in A^{\mathbb{N}}$ and $x \in \bigcap_n B_n$. $\square$

∘ $\mathbb{R}$ is (metric) complete.

Proof. We will use that $\mathbb{R}$ is order-complete, i.e. every set $A \subseteq \mathbb{R}$ bdd above has a supremum ($\iff$ every set $A \subseteq \mathbb{R}$ bdd below has an infimum).
A closed ball in $\mathbb{R}$ is just a closed bdd interval.
Let (I_n) be a decreasing sequence of closed

bdd intervals (and we don't even need that $\mathrm{diam}(I_n) \to 0$).
Let $I_n = [a_n, b_n]$, then (a_n) is increasing & bdd by b_0 & (b_n) is decreasing and bdd by a_0. Then



let $a := \sup\{a_n\}$ & $b := \inf\{b_n\}$.
The interval $\emptyset \neq [a,b] \subseteq \bigcap_n [a_n, b_n]$. □

Cor. The sequence $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ has a limit, $\forall x_0 < x_1$.

$\boxed{\text{HW}}$ Find it.

Proof. Contractive $\Rightarrow$ Cauchy $\Rightarrow$ convergent. □

Obs. A subset Y of a complete metric space (X,d) is closed $\Leftrightarrow$ it's complete with resp. to d.

Proof. $\Rightarrow$. Any Cauchy seq. $(y_n) \subseteq Y$ converges in X to a limit x, but then $x \in Y$ because Y is closed.

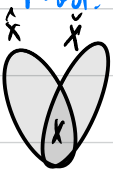
$\Leftarrow$. For any $x \in \overline{Y}$, $\exists$ seq. $(y_n) \subseteq Y$ converging to x. Thus, (y_n) is Cauchy hence has a limit $y \in Y$. But limits are unique in metric spaces, so $x = y \in Y$. □

Metric completion. Let (X,d) be a metric space. A completion of

X is a complete metric space $(\hat{X}, \hat{d})$ s.t. X embeds isometrically into $\hat{X}$ and the image is dense. Identifying X with its image, we may assume WLOG (Without Loss Of Generality) that $X \subseteq \hat{X}$ and $\hat{d}|_X = d$.

<u>Uniqueness</u> (up to isometric isomorphism). If $(\hat{X}, \hat{d})$ and $(\check{X}, \check{d})$ are completions of X then $\exists$ bijective isometry $f: \underset{\underset{X}{\cup\!\!\!\!\!\ \ \ \subseteq}}{\hat{X} \to \check{X}}$ fixing X pointwise, i.e. $f|_X = id_X$.

Proof. Because X is dense in both $\hat{X}$ and $\check{X}$, its closure in $\hat{X}$ is $\hat{X}$ and in $\check{X}$ is $\check{X}$. Define $f: \hat{X} \to \check{X}$ as follows: For each $\hat{x} \in \hat{X}$ choose a seq. $(x_n) \subseteq X$ converging to $\hat{x}$. Then (x_n) is Cauchy in $\hat{X}$ hence also Cauchy in $\check{X}$ because f is an isometry and identity on X.



By the completeness of $\check{X}$, (x_n) has a limit $\check{x} \in \check{X}$, and we define $f(\hat{x}) := \check{x}$.

$\boxed{\text{HW}}$ (a) Prove that f is well-defined, i.e. doesn't depend on the choice of (x_n).

(b) Prove that f is a bijective isometry. □

Existence (of completion). Let (X,d) be a metric space. Let

$$\mathrm{Cauchy}(X) \subseteq X^{\mathbb{N}} \text{ be the set of Cauchy sequences in } X.$$

Define for two $(x_n),(y_n) \in \mathrm{Cauchy}(X)$,

$$D\big((x_n),(y_n)\big) := \lim_n d(x_n,y_n).$$

$\boxed{\text{HW}}$ Prove that $\big(d(x_n,y_n)\big)$ is a Cauchy sequence in $\mathbb{R}$, hence the limit exists.

This D is a pseudo-metric, so we take $\hat{X} := \mathrm{Cauchy}(X)/{\sim_D}$ and get a metric space $(\hat{X},\hat{d})$. For $(x_n) \in \mathrm{Cauchy}(X)$, let $[x_n]$ denote its $\sim_D$-equivalence class.

We embed X into $\hat{X}$ by

$$x \mapsto [x,x,x,\dots].$$

This is clearly an isometry.

We show that the constant sequence from X are dense in $\hat{X}$: Let $[x_n] \in \hat{X}$ and $\varepsilon>0$. Then because (x_n) is Cauchy, $\exists\, n$ s.t. $\operatorname{diam}(\{x_n, x_{n+1}, x_{n+2},\dots\}) < \varepsilon$ hence

$$\hat{d}\big([x_n,x_n,x_n,\dots],[x_0,x_1,x_2,\dots,x_n,x_{n+1},\dots]\big) < \varepsilon.$$

It remains to show that $\hat{X}$ is complete.

Note: Every Cauchy seq. (x_n) is equivalent to any of its subsequence (x_{n_k}), i.e. $[x_k]=[x_{n_k}]$. (By Cauchy-ness.)

To prove completeness, let $(\hat{x}^k) \subseteq \hat{X}$ be a $\hat{d}$-Cauchy sequence, where $\hat{x}^k = [x^k_n]$. Moving to subsequences if needed (HW)

$$
\begin{array}{llllllll}
 & \hat{x}^1 : & \boxed{x^1_1} & x^1_2 & x^1_3 & x^1_4 & x^1_5 & \dots \\
k & \hat{x}^2 : & x^2_1 & \boxed{x^2_2} & x^2_3 & x^2_4 & x^2_5 & \dots \\
 & \hat{x}^3 : & x^3_1 & x^3_2 & \boxed{x^3_3} & x^3_4 & x^3_5 & \dots \quad < \frac{1}{k} \\
n & \hat{x}^4 : & x^4_1 & x^4_2 & x^4_3 & \boxed{x^4_4} & x^4_5 & \dots \\
 & & \vdots & & & & &
\end{array}
$$

we may assume WLOG that $\forall k$:

(i) $\forall n \geq k \quad d(x^n_n, x^k_n) < \frac{1}{k}$.

(ii) $\forall m \quad d(x^k_k, x^k_{k+m}) < \frac{1}{k}$.

Let $\hat{x} := [x^n_n]$.

Claim: (x^n_n) is Cauchy, so $\hat{x} \in \hat{X}$.

Proof. Indeed, $d(x^k_k, x^{k+m}_{k+m}) \leq \underbrace{d(x^k_k, x^k_{k+m})}_{< \frac{1}{k}} + \underbrace{d(x^k_{k+m}, x^{k+m}_{k+m})}_{< \frac{1}{k}} \to 0$ $\square$

Claim. $\hat{x}^k \to \hat{x}$.

Proof. $\hat{d}(\hat{x}^k, \hat{x}) = D((x^k_n), (x^n_n)) = \lim_n d(x^k_n, x^n_n) < \frac{1}{k}$

$\to 0$ as $k \to \infty$. $\square$

$\boxtimes$